Exotic similarity solutions with power-law tails

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Abstract

The diffusion equation is known to have the similarity solutions of the form, $\theta^{\nu} u(\theta x, \theta^2 t) = u(x,t)$ $(\theta \neq 0)$. While we usually deal with those similarity solutions that tends rapidly to constant values for large $|x|$, we evoke the existence of uncountably many other exotic similarity solutions having long tails as $u(x,t) \sim |x|^{-\nu}$ for large $|x|$. While these solutions are often ignored by physical reasons, their existence is worth bearing in mind for the mathematical consistency on the one hand, but also for possible experimental realizations on the other hand. We present an example in the context of slow relaxation of gel accompanying the permeation of solvent.

1 Introduction – Similarity solutions of diffusion equation

The diffusion phenomenon of a field $u(x,t)$ in one spatial dimension is described by the diffusion equation,

$$\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial x^2}\right) u(x,t) = 0, \tag{1}$$

where D is the diffusion constant. This equation has the most basic solution,

$$u_0(x,t) = \frac{M_0}{\sqrt{4\pi D t}} e^{-\frac{x^2}{4Dt}}, \tag{2}$$

where the "mass" M_0 is given by $M_0 = \int_{-\infty}^{+\infty} u(x',0)dx'$. Fig.1 shows the profiles of u_0 for $Dt = 1$, 4 and 9 with $M_0 = 1$. As is seen from the figure, this solution satisfies the property of so-called *dynamical similarity*,[1] $\theta\, u_0(\theta x, \theta^2 t) = u_0(x,t)$ for arbitrary $\theta (\neq 0)$. Similarity solutions are found also in nonlinear processes

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[2]. For the diffusion equation Eq.(1) we can generate many other similarity solutions from $u_0(x,t)$ through the spatial derivations, $\partial^p u_0(x,t)/\partial x^p$ with p being positive integer. They takes the form,

$$u_p(x,t) \equiv \frac{M_p}{(Dt)^{\frac{p+1}{2}}} \phi_p(\frac{x}{\sqrt{Dt}}) \tag{3}$$

with M_p being a constant and $p \geq 1$ being integer. Here

$$\phi_0(s) = \frac{e^{-s^2/4}}{\sqrt{4\pi}}, \quad \phi_p(s) = \frac{d^p \phi_0(s)}{ds^p}, \tag{4}$$

are the scaling function for the scaled variable $s = x/\sqrt{Dt}$, and similarity property of $u_p(x,t)$ is summarized as $\theta^{p+1}\, u_p(\theta x, \theta^2 t) = u_p(x,t)$. We note that all these $u_p(x,t)$ decays to zero with $|x|$ faster than any inverse power of $|x|$.

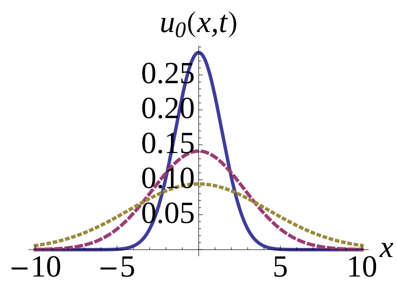

Figure 1: (color online) The most basic similarity solution of the diffusion equation, $u_0(x,t)$ (see (2)) is shown as function of x with $Dt = 1$(solid curve), $Dt = 4$ (dashed curve), $Dt = 9$ (dotted curve) for $M_0 = 1$. The solutions decay with more rapidly than any power low in x.

The fundamental solution, $u_0(x,t)$, together with the above similarity solutions $\{u_p(x,t)\}$ are in a sense complete set for the diffusion equation Eq.(1). First, from any initial distribution $u(x,0$ that decays rapidly enough at $x \to \pm\infty$, the solution $u(x,t)$ asymptotically approaches to the solution $u_0(x-x_0, t+t_0)$ with x_0 and $t_0 (\geq 0)$ being a constant. Secondly any square integrable initial distribution $u(x,0)$ can be decomposed as the linear superposition of $\{u_p(x,t_0)\}$ with t_0 common to all non-negative integers p and M_p being properly chosen. It is because the system $\{u_p(x,t_0)\}$ is a complete basis for the square integrable functional space. It, therefore, assures that the diffusion process starting from any square integrable initial condition $u(x,t)$ is representable as the sum of similarity solutions, $\{u_p(x,t)\}$.[3] In short, Eqs.(3) and (4) capture the similarity properties of the diffusion equation if we limit ourselves to the spatially rapidly decaying functions.

2 'Exotic' similarity solutions with long tail

2.1 Reason for the exotic solutions

Eqs.(3) and (4) are, however, not the all possible similarity solutions of the diffusion equation. From the mathematical consistency view point, we notice that the second order ordinary differential equation that $\phi_p(s)$ should obey has two independent solutions under the boundary conditions $\phi_p(s) \to 0$ for $|s| \to \infty$, while Eq.(4) gives only one of them. The ordinary differential equation for $\phi_p(s)$ is found by substituting $u_p(x,t)$ defined by Eq.(3) into the diffusion equation Eq.(1) as $u = u_p$. The result is

$$\phi_p'' + \frac{s}{2}\phi_p' + \frac{p+1}{2}\phi_p = 0. \tag{5}$$

While the mathematicians have developed long time ago a method to find general similarity solutions [4], its implication was not largely recognized in other fields. From physical point of view, we may wonder if there are similarity solutions of the diffusion process that start from delocalized initial states having long tail like $u(x,0) \sim x^{-\alpha}$ with some positive constant α. As we will discussed later some experimental setup may allow to prepare such initial states. Again, while the mathematicians have discussed the long time asymptotics with algebraic tails [5], such studies have not been well-known in other fields.

2.2 Elementary derivation of some exotic solutions for integer p

Below we describe the class of similarity solutions other than those based on the Gaussian function ($\phi_0(s)$ of Eq.(4)) and show explicitly for p integer that those 'exotic' solutions has the power-law tails for $|s| = |x|/\sqrt{Dt} \gg 1$. The extension to the case with p non-integer will be discussed in the next subsection.

We first look for a new type of solutions of Eq.(5) with $p = 0$, which we denote as $\tilde{\phi}_0(s)$. The problem is to solve generally

$$\left(\frac{d^2}{ds^2} + \frac{s}{2}\frac{d}{ds} + \frac{1}{2}\right)\tilde{\phi}_0(s) = 0, \tag{6}$$

under the boundary conditions, $\tilde{\phi}_0(\pm\infty) = 0$. We immediately find the first integral of Eq.(6):

$$\left(\frac{d}{ds} + \frac{s}{2}\right)\tilde{\phi}_0(s) = C, \tag{7}$$

with C being the constant of integral. While $\phi_0(s)$ of Eq.(4) is the solution of homogeneous equation, i.e. with $C = 0$, the special solution with $C \neq 0$ gives qualitatively different solution. Since $\phi_0(s)$ is symmetric in s, we can eliminate this homogeneous solution by imposing $\tilde{\phi}_0(0) = 0$. With choosing $C = (4\pi)^{-\frac{1}{2}}$, the special solution is written as follows:

$$\tilde{\phi}_0(s) \equiv \frac{e^{-\frac{s^2}{4}}}{\sqrt{4\pi}}\int_0^s e^{\frac{w^2}{4}}dw$$

$$= \quad \phi_0(s)\int_0^s e^{\frac{w^2}{4}}dw. \tag{8}$$

While the case of $C \neq 0$ is disregarded in the usual argument of similarity solutions, this case is compatible with the vanishing boundary conditions, $\tilde{\phi}_0(\pm\infty) = 0$ [6] Apparently, $\tilde{\phi}_0(s)$ behaves in qualitatively different manner from the Gaussian function $\phi_0(s)$ in Eq.(4), because the former is multiplied by a rapidly growing integral. More concretely, $\tilde{\phi}_0(s)$ has a power low asymptote, $\sim s^{-1}$, for $|s| \gg 1$. For $s > 0$, for example, we can introduce $v = s - w$ in the integral in Eq.(8) and find

$$\begin{aligned}\tilde{\phi}_0(s) &= \frac{1}{\sqrt{4\pi}}\int_0^s e^{-\frac{sv}{2}+\frac{v^2}{4}}dv \\ &\simeq \frac{1}{\sqrt{4\pi}}\int_0^s e^{-\frac{sv}{2}}dv \simeq \frac{1}{\sqrt{\pi}\,s}.\end{aligned} \tag{9}$$

Once we find the new solution for $p = 0$, the new solution for Eq.(5) with any non-negative integer p is found as

$$\tilde{\phi}_p(s) \equiv \frac{d^p\tilde{\phi}_0(s)}{ds^p}. \tag{10}$$

In Fig.2 we show $\tilde{\phi}_0$, $\tilde{\phi}_1$ and $\tilde{\phi}_2$. We thus identified countably many exotic similarity solutions of Eq.(1), which we denote as $\tilde{u}_p(x,t)$. They obey the same similarity relation as Eq.(3), that is,

$$\tilde{u}_p(x,t) = \frac{\tilde{M}_p}{(Dt)^{\frac{p+1}{2}}}\tilde{\phi}_p(\frac{x}{\sqrt{Dt}}), \tag{11}$$

where $\tilde{M}_p$ is constant. Because $\tilde{\phi}_0(s) \simeq (\sqrt{\pi}s)^{-1}$ for $|s| \gg 1$, we find that $\tilde{\phi}_p$

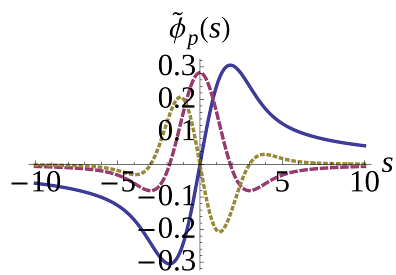

Figure 2: (color online) Some scaling functions $\tilde{\phi}_p(s)$ in Eq.(11) vs s with $p = 0$ (solid curve), $p = 1$ (dashed curve), $p = 2$ (dotted curve). These are 'exotic' due to their algebraic tail, $\tilde{\phi}_p(s) \sim s^{-(p+1)}$ for $|s| \gg 1$.

behaves as

$$\tilde{\phi}_p(s) \simeq \frac{(-1)^p(p-1)!}{\sqrt{\pi}s^{p+1}}, \quad \text{for } |s| \gg 1. \tag{12}$$

The exponent $-(p+1)$ in Eq. (12) is understandable if we ignore ϕ_p'' in Eq.(5). Now if we combine Eq.(11) and Eq.(12), we find that the long tail $\sim s^{-(p+1)}$ of $\tilde{\phi}_p(s)$ causes a long spatial tail of $\tilde{u}_p(x,t)$ which is *time-independent* for $|x| \gg \sqrt{Dt}$:

$$\tilde{u}_p(x,t) \simeq \frac{(-1)^p (p-1)! \tilde{M}_p}{\sqrt{\pi}} \frac{1}{x^{p+1}}. \tag{13}$$

We can see this fact in Fig.3 where the graphs of $\tilde{u}_0(x,t)$ vs x are shown for $Dt = 1, 4$ and 9. The asymptotically time independent tail implies that this type

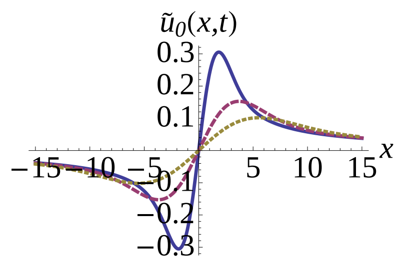

Figure 3: (color online) An 'exotic' similarity solution of the diffusion equation, $\tilde{u}_0(x,t)$ (see (11) and (8)) is shown as function of x with $Dt = 1$(solid curve), $Dt = 4$ (dashed curve), $Dt = 9$ (dotted curve) for $\tilde{M}_0 = 1$. We notice that $\tilde{u}_0(x,t)$ has a time-independent long low tails, $\sim x^{-1}$ for $|x| \gg \sqrt{Dt}$.

of similarity behavior corresponds to physically different situation from Gaussian relaxation. The diffusion can effectively modify the distribution $u(x,t)$ only up to the range of $\sim \sqrt{Dt}$ from its center but cannot modify the long spatial tail.

2.3 The case of non-integer p

We search for the similarity solutions of the form Eq.(3). In other words, we search for $\phi_p(s)$ that satisfies Eq.(5), under the boundary conditions, $\phi_p(\pm\infty) = 0$. The general solution of Eq.(5) is given as a combination of confluent hypergeometric functions multiplied by $e^{-\frac{s^2}{4}}$. Using the fact that Eq.(5) is invariant under the inversion $s \leftrightarrow -s$, we can find an 'even' solution under the inversion, and an 'odd' solution which is antisymmetric under the inversion. The former, or symmetric, solution satisfying $\phi_p^{(\mathrm{s})}(0) = 1$, and $\phi_p^{(\mathrm{s})\prime}(0) = 0$ is

$$\phi_p^{(\mathrm{s})}(s) = e^{-\frac{s^2}{4}} {}_1F_1\left(-\frac{p}{2}, \frac{1}{2}, \frac{s^2}{4}\right)$$

and the latter, or antisymmetric, solution satisfying $\phi_p^{(\mathrm{a})}(0) = 0$, and $\phi_p^{(\mathrm{a})\prime}(0) = 1$ is

$$\phi_p^{(\mathrm{a})}(s) = s\, e^{-\frac{s^2}{4}} {}_1F_1\left(\frac{1-p}{2}; \frac{3}{2}; \frac{s^2}{4}\right),$$

where ${}_1F_1(a;b;z)$ is the confluent hypergeometric function of the first kind [7]. These two solutions constitute the two independent solutions of Eq.(5). Important is that, as long as p is non integer, both these solutions have long tails $\sim s^{-(1+p)}$ for $|s| \gg 1$. It is only when p in Eq.(3) tends to a non-negative integer that power-low tails of one of the above solutions disappear. The above power-law tails associated to the dynamical similarity is related neither to the diffusion on a fractal space [9] no to any super-diffusive Brownian motion. In short, a simple one-dimensional diffusion equation has uncountably many similarity solutions with power-law tails.

3 Similarity solution of three-dimensional diffusion with power-law tail

The similarity solution with long tails like $\tilde{u}_0(x,t)$ in Fig.3 might seem to be non-realizable in experiments. Nevertheless, we propose a possible realization of similarity solution with power-law tail in the diffusion phenomena in three-dimensional gel system. Gel is a dilute polymer network saturated with solvent, and the solvent can slowly permeate through this network under a pressure gradient. It has been known that, when the gel network relaxes to the homogeneous state through a quasi-static balance between the elastic restoring force of the gel network (osmotic stress) and the viscous friction force between the network and the solvent (permeation resistance), the local displacement vector of the gel $\vec{U}(\vec{r},t)$ as functions of position $\vec{r}$ and time t obeys the three-dimensional vector diffusion equation [10]:

$$\left[f\frac{\partial}{\partial t} - \mu\Delta - (K+\frac{\mu}{3})\vec{\nabla}\vec{\nabla}\cdot\right]\vec{U} = 0, \tag{14}$$

where f, μ, K are, respectively the friction coefficient between the gel network and solvent, the shear modulus and bulk (osmotic) modulus of the gel [11], and $\Delta \equiv \vec{\nabla}\cdot\vec{\nabla}$ is three-dimensional laplacian operator. Eq.(14) is sightly related to the diffusion equation when we notice that the linearized local density deviation $\delta\rho$ of the gel's density relative to its uniform reference density, ρ_0, is related to $\vec{U}$ by $\delta\rho/\rho_0 = -\vec{\nabla}\cdot\vec{U}$. The density deviation satisfies the scalar equation,

$$\left(\frac{\partial}{\partial t} - D\Delta\right)\delta\rho = 0 \tag{15}$$

with the (cooperative) diffusion constant $D = (K + 4\mu/3)/f$. It is expected since the gel's density must be conserved. Hereafter, we focus on the spherically symmetric case and put input the radial displacement field $U^{(\mathrm{r})}(r,t)$ by $\vec{U}(\vec{r},t) = \hat{r}U^{(\mathrm{r})}(r,t)$, where r and $\hat{r}$ are radial the distance and the normalized radial vector, respectively. Then $U^{(\mathrm{r})}(r,t)$ obeys

$$\left(\frac{\partial}{\partial t} - D\left[\frac{\partial^2}{\partial r^2} - \frac{2}{r^2}\right]\right)[r\,U^{(\mathrm{r})}(r,t)] = 0. \tag{16}$$

We note that, in terms of the density deviation, $-\delta\rho(r,t)/\rho_0 = (\partial/\partial r + 2/r)$ $U^{(r)}(r,t)$, this equation is again reduced to the one-dimensional diffusion equation for $r\rho(r,t)$ for $0 \leq r < \infty$ [12],

$$\left(\frac{\partial}{\partial t} - D\frac{\partial^2}{\partial r^2}\right)[r\,\delta\rho(r,t)] = 0. \tag{17}$$

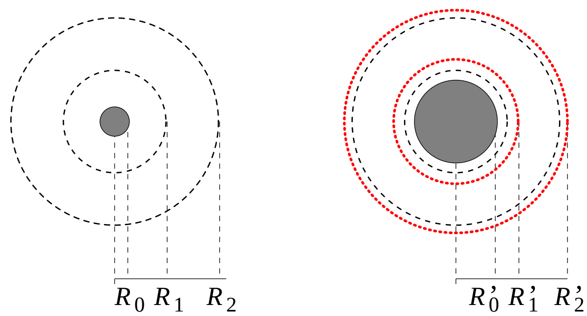

Figure 4: (color online) Schematic representation of proposed setup of gel experiment. (Left) An isotropic gel as a reference state for $t < 0$. (Right) The deformed state of gel just after the injection of solvent into the shaded region (of volume $V_0 = 4\pi R_0^3/4$ originally, and $(1+\epsilon)V_0 = 4\pi {R_0'}^3/4$ after the injection) at $t = 0+$. While the injection swells the shaded region, it also causes quasi-instantaneous and quasi-incompressible displacements outside of this region. For example, the material point at the radii R_i ($i = 1,2$: dashed circles) are displaced to $R_i' = R_i + U^{(\mathrm{r})}(R_i, 0+)$ (dotted circles), respectively, with $4\pi R_i^3/3 + \epsilon V_0 = 4\pi {R_i'}^3/3$. The last relation approximately gives Eq.(19).

Experimentally we may consider the following setup (see Fig.4: We prepare a gel in the homogeneous reference state with $\vec{U} \equiv 0$. Then we suddenly inject the solvent into a localized volume, $V_0 = (4\pi/3)R_0^3$, around the origin. If the injection of solvent is done very quickly, this causes, on the one hand, almost uniform dilatation of gel over this volume,

$$U^{(\mathrm{r})}(r,0) = \frac{\epsilon}{3}r, \qquad 0 \leq r \leq R_0, \tag{18}$$

where ϵV_0 is the total injected volume of the solvent. On the other hand, this injection causes the outward displacement of the gel outside this region, $U^{(\mathrm{r},0)}$ for $R_0 < r < \infty$. Because the permeation of solvent through the network is very slow, the initial displacement is such that the gel is thus radially 'pushed off' in the region outside of V_0 while the network is incompressible: $\vec{\nabla} \cdot \vec{U}(\vec{r},0) = (\partial/\partial r + 2/r)U^{(r)}(r,0) = 0$. Knowing the symmetry condition, $U^{(\mathrm{r})}(0;t) = 0$, and the volume dilatation at the core region, ϵV_0, we find

$$4\pi r^2 U^{(\mathrm{r})}(r,0) = \epsilon V_0, \qquad R_0 < r < \infty. \tag{19}$$

We then find the initial displacement field realizing a power-law tail, $\sim r^{-2}$. In terms of the gel's density, the gel is almost uniformly diluted inside the volume V_0 so that $\delta\rho(r,0)/\rho_0 = -\epsilon < 0$, while outside of V_0 the gel keeps its density, $\delta\rho(r,0) = 0$.

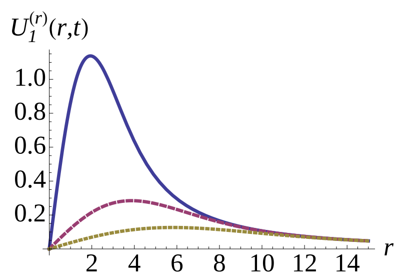

Figure 5: (color online) A spherically symmetric similarity solution of three dimensional vector diffusion equation, $U^{(\mathrm{r})}_{p=1}(r,t)$ (see Eq.(20)) is shown vs the radius r with $Dt = 1$(solid curve), $Dt = 4$ (dashed curve), $Dt = 9$ (dotted curve). The vertical scale has been arbitrarily fixed. The solution $U_1^{(\mathrm{r})}(r,t)$ shows a time-independent tail, $\sim r^{-2}$ for $r \gg \sqrt{Dt}$.

To know the time evolution of $U^{(\mathrm{r})}(r,t)$, we calculated in Appendix the class of similarity solutions of Eq.(16) under the similarity hypothesis,

$$U^{(\mathrm{r})}(r,t) = U_p^{(\mathrm{r})}(r,t) \equiv \frac{1}{(Dt)^{\frac{p+1}{2}}} \psi_p\left(\frac{r}{\sqrt{Dt}}\right), \tag{20}$$

where $p = 1$ corresponds to the gel's relaxation from the initial condition mentioned above. The radial displacement field is given by $U_1^{(\mathrm{r})}(r,t) = \psi_1(r/\sqrt{Dt})/(Dt)$. In Fig.5 we show $U^{(\mathrm{r})}_{p=1}(r,t)$ at $Dt = 1, 4$ and 9 with an arbitrarily chosen value of ϵV_0. In the figure we see an asymptotically time-independent power-law tail,

$$U_1^{(\mathrm{r})}(r,t) \sim \frac{\text{const.}}{r^2}, \qquad \frac{r}{\sqrt{Dt}} \gg 1, \tag{21}$$

while the zone of dilatation broadens gradually as $\sim \sqrt{Dt}$. For this system the density deviation $\delta\rho(r,t)/\rho_0$ can be calculated from $U_1^{(\mathrm{r})}(r,t)$ and found to be (see Appendix)

$$\frac{\delta\rho(r,t)}{\rho_0} = -\epsilon\, e^{-\frac{r^2}{4Dt}} \frac{V_0}{(4\pi Dt)^{3/2}}.$$

The right hand side shows that the radius of the dilated region grows from $\sim V_0^{1/3}$ to $\sim (Dt)^{1/2}$ while outside this region the gel remains at the original density, ρ_0. In other words, if we observe only the density profile, the persistence of time-independent tail $\sim r^{-2}$ in $U_1^{(\mathrm{r})}(r,t)$ is completely masked.

4 Conclusion

We discussed the exotic similarity solutions of diffusion and related equations that exhibit power-law tails. While these solutions are usually ignored, there are both mathematical and physical reasons to ask about them. The radial displacement field in yes, $U^{(\mathrm{r})}(r,t)$, is an observable quantity, and such observable shows a similarity relation with long spatial tail.

Acknowledgments

We thank Antoine Fruleux at ESPCI for the discussion.

A Similarity solution of Eq.(16)

We substitute the form of Eq.(20), that is,

$$U^{(\mathrm{r,t})} = U_p^{(\mathrm{r})}(r,t) \equiv \frac{1}{(Dt)^{\frac{p+1}{2}}}\psi_p\left(\frac{r}{\sqrt{Dt}}\right),$$

into Eq.(16). This yields the condition for $\psi_p(s)$:

$$2s^2\psi_p''(s) + s(s^2+4)\psi_p'(s) + ((p+1)s^2-4)\psi_p(s) = 0 \quad (s>0). \tag{22}$$

To this equation we must impose, in addition to the condition $\psi_p(\infty)=0$, the condition $\psi_p(0)=0$ so that the displacement vector field $\vec{U}$ is regular at the origin, that is, $\vec{U}(\vec{r},t) = c(t)\,\vec{r} + \mathcal{O}(r^3)$ for $r \downarrow 0$ with $c(t)$ a function of time. Under these boundary conditions, the solution of Eq.(22) is written in terms of the confluent hypergeometric function ${}_1F_1(a;b;z)$. If we impose $\psi_p'(0)=1$ just for the normalization purpose, the solution reads

$$\psi_p(s) = s \times {}_1F_1\left(1+\frac{p}{2};\frac{5}{2};-\frac{s^2}{2}\right). \tag{23}$$

This solution behaves as $\psi_p(s) \simeq s$ for $0 \le s \ll 1$ and $\psi_p(s) \sim s^{-(p+1)}$ for $s \gg 1$. For $p=1$ which we are interested in, the first derivative, $\psi_1'(s)$ can be explicitly given by

$$\psi_1'(s) = -\frac{12\sqrt{\pi}}{s^3}\mathrm{Erf}\left(\frac{s}{2}\right) + 3(1+\frac{4}{s^2})e^{-\frac{s^2}{2}}, \tag{24}$$

where $\mathrm{Erf}(z) \equiv (2/\sqrt{\pi})\int_0^z e^{-u^2/2}du$ is the error function. The right hand side of Eq.(24) tends to $-12\sqrt{\pi}/s^3$ for $s \uparrow \infty$. We can also find $\psi_1(s)$ by integrating the above formula from $\psi_1(0)=0$ to s (not shown). Using ψ_1 and ψ_1', we find

$$-\psi_1'(s) + 2\psi_1(s)/s = -3e^{-s^2/2}.$$

With $s = r/\sqrt{Dt}$, the left hand side of the last equation is proportional to the density deviation, $\delta\rho(r,t)/\rho_0$ times $(Dt)^{3/2}$.

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[2] For example, the Burgers equation, $\frac{\partial v}{\partial t} + v\frac{\partial v}{\partial x} - D\frac{\partial^2 v}{\partial x^2} = 0$, also has similarity solutions for $v(x,t)$ because this equation can be transformed into Eq.(1) by the well-known Cole-Hopf transformation, $v = -\frac{2D}{u}\frac{\partial u}{\partial x}$.

[3] The set of functions $\{\phi_p(s)\}_{p=0}^{\infty}$ span the same functional space as $\{s^p\phi_0(s)\}$, which in turn span the same space as the quantum-mechanical basis set of a harmonic oscillator, $\{H_p(s)\phi_0(s)\}$, where $H_p(s)$ is s-th Hermite polynomial. There are infinitely many different ways of decomposition because we can choose arbitrarily shift the the center x_0 and the initial time t_0 of this basis set.

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[5] Guillaume van Baalen, Nikola Popović, and C. Eugene Wayne, SIAM J. Math. Anal., **39**, 1951–1977 (2007).

[6] This is in contrast with the case when we impose the different asymptotic boundary values of the diffusion equation. In such case the integral constant corresponding to C is forced to adjust: Under the boundary conditions $\lim_{x\to\pm\infty} u(x,t) = u_\pm$, the similarity solution of (1) is $u(x,t) = (u_+ + u_-)/2 + (u_+ - u_-)/\sqrt{4\pi D} \times \int_0^s e^{-\frac{u^2}{4D}} du|_{s=(x-x_0)/\sqrt{Dt}}$, with x_0 being a constant.

[7] For the confluent hypergeometric function, see for example, Milton Abramowitz and Irene A. Stegun, *Handbook of Mathematical Functions: with Formulas, Graphs, and Mathematical Tables*, Ser. Dover Books on Mathematics (Dover, London, 1965), p. 505.

[8] By combining the even and odd solutions belonging to the same index p, we can even construct similarity solutions that decays either only for $s \to +\infty$ or for $s \to -\infty$.

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[11] Pierre Gilles de Gennes, *Scaling Concepts in Polymer Physics*, (Cornell University Press, Ithaca, NY,1979).

[12] However, the general solution of Eq.(17) contains the fictitious solutions which are not compatible with Eq.(16).